RAPID COMMUNICATIONS

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Spatial periodic synchronization of chaos in coupled ring and linear arrays of chaotic systems

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Dynamic behaviors of coupled ring and linear arrays of unidirectionally coupled Lorenz oscillators are studied numerically. It is found that the chaotic rotating waves generated from the ring propagate with spatial periodic synchronization along the linear array, that is to say, two chaotic oscillators in the linear array are synchronized if the number of oscillators (spatial distance) between them is a multiple of oscillator number in the ring. Numerically it is shown that the stabilities of the synchronized states are enhanced by chaos, and degraded when the oscillators are far from the ring.

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Chaos synchronization of coupled nonlinear systems [1–4] is ubiquitous in nature and science and recently has attracted great interest in chaos study [5–7]. Several types of synchronization, such as generalized synchronization [8], phase synchronization [9], lag synchronization [10], and so on, are shown to exist in different geometries of oscillators coupling. But until now much study on coupled chaotic oscillators has focused mainly on the instability of the uniform synchronous state, in which either all oscillators synchronize to the drive or all of them do not synchronize in the long time region [3,4,13–15]. Ring geometries have been used extensively in physiological and biochemical modeling studies [11,12]. Ring and linear arrays are two main coupling models in the study of chaos synchronization [4,13–20] since their potential applications in communication [21] and neural process [16]. In this Rapid Communication we couple a ring and a linear array and numerically found that the chaos of contiguous units of the linear array do not synchronize, but that of noncontiguous ones synchronize with spatial period.

A scheme of our coupling geometry is shown in Fig. 1, in which linear array is driven by the circular array of unidirectionally coupled Lorenz oscillators. The arrow indicates an oscillator drives the one closely behind it, that is to say, chaotic waves unidirectionally propagate in the arrays. All oscillators both in the ring and in the linear array are identical and connected unidirectionally through variable x with the same coupling strengths. With these considerations we have the following evolution equations for the system:

$$\dot{x}_j=\sigma(y_j-x_j),$$
$$\dot{y}_j=R[\alpha\overline{x_j}+(1-\alpha)x_j]-y_j-x_jz_j,$$
$$\dot{z}_j=x_jy_j-bz_j, \tag{1}$$

($j=1,\ldots,N;1',\ldots,N'$) where $\overline{x_j}=x_{j-1}$ for $j\neq 1$, the coupling strength α allows one to control the stability of the connection, and $0\leqslant\alpha\leqslant 1$. The boundary conditions enter through $\overline{x_j}$, which takes the value $\overline{x_{1'}}=x_{N'}$ for circular arrays, while for linear arrays it is $\overline{x_1}=x_{N'}$ in our system. The oscillators are labeled by $k'=1',2',3',\ldots,N'$ in the ring, and $l=1,2,3,\ldots,N$ in the linear array. The size of the ring is $m=N'$.

In the coupling geometry the ring can be treated as the external drive and the linear array as the response system. In the numerical studies we focus mainly on the special case, where $m=3$, and $N\rightarrow\infty$. The parameters, σ, R, and b which have the usual meaning, are chosen in the chaotic region of the isolated Lorenz oscillator, in our case, $\sigma=20$, $b=2.5$, and $R\geqslant 28$. Taking $\alpha=1$ and $R=28$ we find that there is a uniform chaotic synchronization in the ring [17]. The ring imposes its behavior to the linear array, and the whole linear array attains the same asymptotic synchronous chaotic behavior as in the ring. For larger values of R up to $R=35$, three different chaotic rotating waves (CRWs) with a $2\pi/3$ phase difference of neighboring oscillators appear in the ring [17,19], see Fig. 2. The inset illustrates the fine detail in which the three CRWs are similar, while the positions and heights of the chaotic wave peaks are different. Any three neighboring oscillators in the linear array respond to the ring and exhibit the same behavior as that in the ring in the long time region. By careful study we found that the noncontiguous oscillators k' and l_0+nm become synchronized with spatial period m ($m=3$ in our special case) in the linear array, here $k'=1',2',3'$; $l_0=1,2,3$; $n=0,1,2,\ldots$ and $k=l_0$, for example, oscillators $1',1,4,7,10,13,\ldots,1+3n$ are all synchronized. Numerical calculations show that stability of the synchronization decreases with increasing n. As an example, we analyze the synchronizations of the oscillators $1'$, and $1,4,7,10,13,\ldots,1+3n$ by using the method of Ref. [2] and writing the linearized evolution equations for ($\delta x_{1-1'}$, $\delta x_{4-1'}$, $\delta x_{7-1'}$, $\delta x_{10-1'}$, $\delta x_{13-1'},\ldots$, $\delta x_{(1+3n)-1'}$)$=$($x_1-x_{1'}$, $x_4-x_{1'}$, $x_7-x_{1'}$, $x_{10}-x_{1'}$, $x_{13}-x_{1'},\ldots,x_{1+3n}-x_{1'}$); here $x_j=(x_j,y_j,z_j)$. Two oscillators, such as x_1 and $x_{1'}$ will synchronize only if $\delta x\rightarrow 0$ as $t\rightarrow\infty$ [2]. Figure 3 exhibits the numerical results ($n=0,\ldots,4$), which shows that the unsynchronized time re-

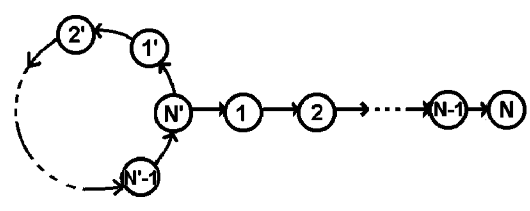

FIG. 1. Geometry of the coupled ring and linear arrays of Lorenz systems, explained in the text.

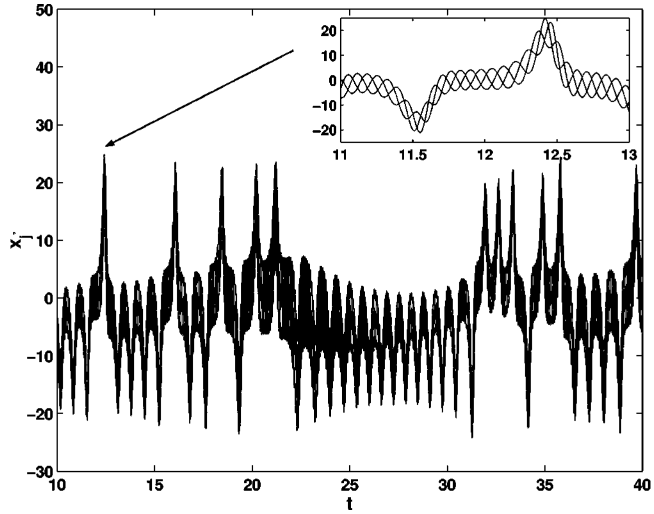

FIG. 2. Diagram of variable $x_{j'}$ vs time t in the coupled ring and linear arrays calculated according Eq. (1), showing the chaotic rotating waves, and the parameters are $\sigma=20$, $b=2.5$, $R=35$, and $\alpha=1$. The inset illustrates the fine detail of the chaotic rotating waves.

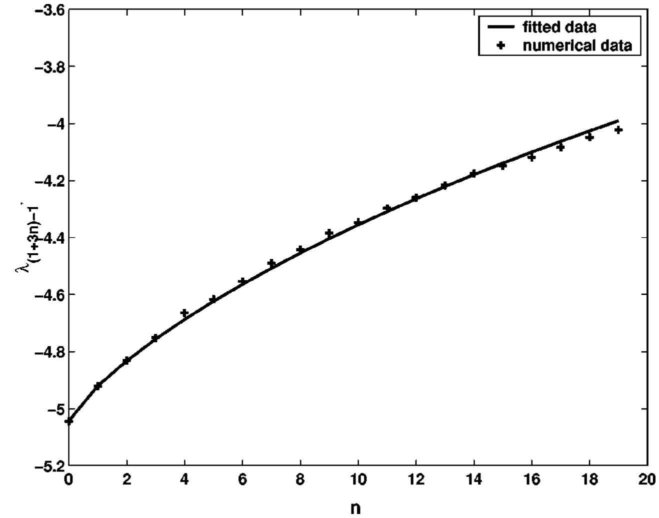

FIG. 4. The diagram of the transverse Lyapunov exponents $\lambda_{(1+3n)-1'}$ vs n in the system. Parameters are the same as those in Fig. 2.

gions $0\rightarrow t_u$ increase with increasing n, that is, the stability of synchronizations becomes more and more weaker down the linear array. The fitted result can be expressed as $t_u = \exp[f(a_1,a_2,a_3,\lambda)]$, where a_1, a_2, and a_3 are dependent on the parameters of Eq. (1). In our case [see Fig. 5(a)], $f(a_1,a_2,a_3,\lambda)=a_1\lambda+a_2 n^{a_3}$; here $a_1\sim 2.0$, $a_2\sim 10.5$, $a_3\sim 0.01$, and λ is the Lyapunov exponent (see below). We have calculated the transverse Lyapunov exponents for $(\delta x_{1-1'}, \delta x_{4-1'}, \delta x_{7-1'}, \delta x_{10-1'}, \delta x_{13-1'}, \ldots, \delta x_{(1+3n)-1'})$ and obtained the fitted formula $\lambda_{(1+3n)-1'} = -c/(n^{\gamma}+d)(c\sim 201.7,\, d\sim 40.0, \gamma\sim 0.80)$. The numerical and fitted results are shown in Fig. 4, whose numerical accuracy is 10^{-2}. The $\lambda_{(1+3n)-1'}$'s increase with increasing n and approach 0 as $n\rightarrow\infty$, then the system becomes unsynchronized completely. The scaling laws about t_u and $\lambda_{(1+3n)-1'}$ are obtained under certain parameters and $m=3$. We should stress that the parameters used in t_u and $\lambda_{(1+3n)-1'}$ are different for different σ, b, R, α, and (or) m. There is no definite relation between them.

We have extensively studied different types of waves that are generated by changing the parameters of Eq. (1), and found that not only CRWs but also quasiperiodic waves and periodic rotating waves (PRWs) can propagate down the linear array with similar results. But synchronization itself is structurally stable in chaotic driving and CRWs can propagate longer than PRWs down the linear array. As an example, the PRWs take a longer time t_u to synchronization than do CRWs (see Fig. 5) for $\alpha=1$, $R=40$, $\sigma=20$, and $b=2.5$.

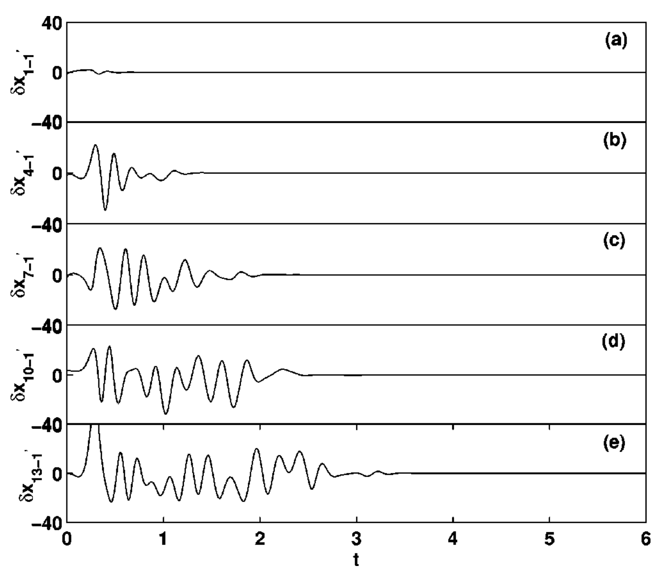

FIG. 3. Representation of δx vs time t in the linear array. (a) $\delta x_{1-1'}$ vs t; (b) $\delta x_{4-1'}$ vs t; (c) $\delta x_{7-1'}$ vs t; (d) $\delta x_{10-1'}$ vs t; (e) $\delta x_{13-1'}$ vs t. Parameters are the same as those in Fig. 2.

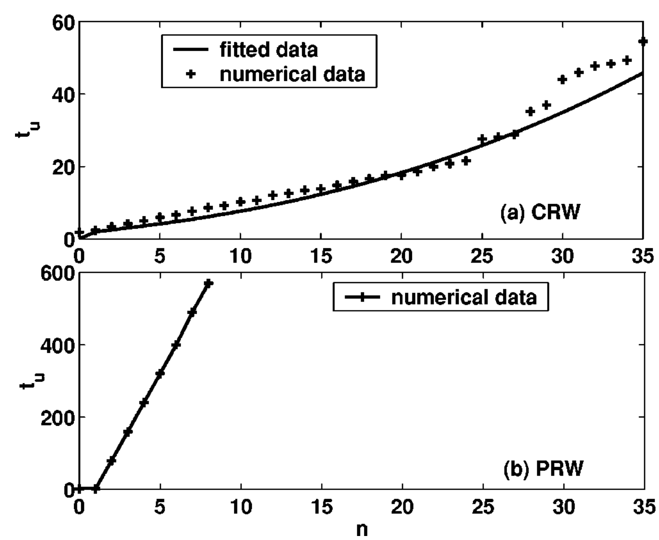

FIG. 5. The relations of time to synchronization t_u with the number of spatial period n, (a) for CRWs, the parameters are the same as those in Fig. 2(b) for PRWs, the parameter same as in (a) except $R=40$.

While the PRWs also can only synchronize in the first few spatial periods, that is, n is small, for example, $n \leqslant 4$ for $\alpha = 1$, $R = 28$, $\sigma = 10$, and $b = 8/3$. Further study shows that spatial periodic synchronization can be observed for arbitrary size of the ring ($m \geqslant 3$), and we have tested this phenomenon up to the size $m = 60$. For a larger ring due to Hopf bifurcation many modes are excited, so other different types of waves appear in the ring. In each of the cases, m different chaotic waves (not always CRWs, there are intermittent chaotic waves, modulated waves, etc.) with a $2\pi/m$ phase difference of neighboring oscillators generated in the ring propagate with spatial period m in the linear array. However, when $m > 3$, the waves generated in the ring are different from that in the ring of $m = 3$ for the same parameters σ, b, R, α. While the forms of the functions $t_u(n)$ and $\lambda_{(l_0+nm)-k'}(n)$ for CRWs are similar to the results given before, the difference is only the parameters in the functions.

Since the coupling affects the spatiotemporal structure, we also carefully studied the evolution of the system with the change of the coupling strength α. The effect of α on the ring is similar to that of R on the single Lorenz oscillator, for example, taking $R = 40$ and decreasing α from 1, the system ($m = 3$) may exhibit a transition from PRWs ($0.90 \lesssim \alpha \lesssim 1$) to CRWs ($0.80 \lesssim \alpha \lesssim 0.90$), then to synchronized chaotic state ($0.20 \lesssim \alpha \lesssim 0.80$). When $\alpha = 0.75$, the global transient chaotic rotating waves appear before they approach the synchronized chaotic behaviors as in the linear array [15]. If we further decrease R and α, we could observe metastable chaos [5] in the system. With decreasing α, the amplitude and frequency of PRWs decrease, and finally they become chaotic waves. The situations are different for $m > 3$, but there are no special interesting cases related to the subject discussed here. The numerical results illustrate that when $\alpha \lesssim 0.10$, the dynamical characteristics of individual Lorenz oscillators dominate the system, and the spatial periodic synchronization of chaos disappears.

In conclusion, we have built up a coupled ring and linear array system, in which chaotic synchronization occurs with spatial period m in the linear array due to the response of the linear array to the ring in which the chaotic rotating waves are generated. The chaos synchronization is degraded when the oscillators in the linear array are far from the ring. This result is correct for other waves. But only the chaotic rotating wave is the most stable wave, that is, chaos can enhance the propagation of the chaotic waves input to the linear array by the ring. In addition, the results shown in the present Rapid Communication can also be found in other coupled oscillators, such as coupled Chua's [23] and Rössler's [24] systems. Our results may have potential applications in neural processes [16], communications [21,22], and information processing systems [13]. How to design chaos synchronizing systems and find their applications in science and technology remains an interesting topic for future research. Another interesting subject is whether there is the noise enhanced propagation [25] in our system.

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